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Research Article

On Lyapunov-Type Inequalities for Two-Dimensional Nonlinear Partial Systems

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We establish a new Lyapunov-type inequality for two nonlinear systems of partial differential equations and the discrete analogue is also established. As application, boundness of the two-dimensional Emden-Fowler-type equation is proved.

1. Introduction

In a celebrated paper of 1893, Liapunov [1] proved the following well-known inequality: if y is a nontrivial solution of

$$y'' + q(t)y = 0, \quad (1.1)$$

on an interval containing the points a and b ($a < b$) such that $y(a) = y(b) = 0$, then

$$4 < (b - a) \int_a^b |q(s)| ds. \quad (1.2)$$

Since the appearance of Liapunov's fundamental paper [1], considerable attention has been given to various extensions and improvements of the Lyapunov-type inequality from

different viewpoints [2–7]. In particular, the Lyapunov-type inequalities for the following nonlinear system of differential equations were given in [8]

$$\begin{aligned}x'(t) &= \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t), \\u'(t) &= -\beta_2(t)|x(t)|^{\beta-2}x(t) - \alpha_1(t)u(t).\end{aligned}\tag{1.3}$$

In this paper, we obtain new Lyapunov-type inequalities for the two-dimensional nonlinear system and discrete nonlinear system, respectively.

2. The Lyapunov-Type Integral Inequality for the Two-Dimensional Nonlinear System

$$\begin{aligned}\frac{\partial^2 x(s,t)}{\partial s \partial t} &= \alpha_1(s,t)x(s,t) + \beta_1(s,t)|u(s,t)|^{\gamma-2}u(s,t), \\ \frac{\partial^2 u(s,t)}{\partial s \partial t} &= -\beta_2(s,t)|x(s,t)|^{\beta-2}x(s,t) - \alpha_1(s,t)u(s,t).\end{aligned}\tag{2.1}$$

We shall assume the existence of nontrivial solution $(x(s,t), u(s,t))$ of the system (2.1), and furthermore, (2.1) satisfies the following assumptions (i), (ii), and (iii):

- (i) $\gamma > 1, \beta > 1$ are real constants;
- (ii) $\beta_1(s,t), \beta_2(s,t) : [s_0, \infty) \times [t_0, \infty) \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions such that $\beta_1(s,t) > 0$ for $(s,t) \in [s_0, \infty) \times [t_0, \infty)$;
- (iii) $\alpha_1(s,t) : [s_0, \infty) \times [t_0, \infty) \rightarrow \mathbb{R}$ is a continuous function.

Theorem 2.1. *Let the hypotheses (i)–(iii) hold. If the nonlinear system (2.1) has a real solution $(x(s,t), u(s,t))$ such that $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ for $(s,t) \in [a,b] \times [c,d]$, and $(\partial u(s,t)/\partial s)(\partial x(s,t)/\partial t) + (\partial u(s,t)/\partial t)(\partial x(s,t)/\partial s)$ and $x(s,t)$ is not identically zero on $[a,b] \times [c,d]$, where $a, b, c, d \in \mathbb{R}$ with $a < b, c < d$, then*

$$2 \leq \int_a^b \int_c^d |\alpha_1(s,t)| dt ds + M^{\beta/\alpha-1} \left(\int_a^b \int_c^d \beta_1(s,t) dt ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2^+(s,t) dt ds \right)^{1/\alpha}, \tag{2.2}$$

where $(1/\alpha) + (1/\gamma) = 1$, $M = \max_{\substack{a < s < b \\ c < t < d}} |x(s,t)|$, and $\beta_2^+(s,t) = \max_{\substack{a < s < b \\ c < t < d}} \{\beta_2(s,t), 0\}$ is the nonnegative part of $\beta_2(s,t)$.

Proof. Since $x(a,t) = x(b,t) = x(s,c) = x(s,d) = 0$ and $x(s,t)$ is not identically zero on $[a,b] \times [c,d]$, we can choose $(\tau, \sigma) \in (a,b) \times (c,d)$ such that $|x(\tau, \sigma)| = \max_{\substack{a < s < b \\ c < t < d}} |x(s,t)| > 0$.

Let $M = |x(\tau, \sigma)| > 0$. Integrating the first equation of system (2.1) over t from c to σ and over s from a to τ , respectively, we obtain

$$\int_a^\tau \int_c^\sigma \frac{\partial^2 x(s, t)}{\partial s \partial t} dt ds = \int_a^\tau \int_c^\sigma \left(\alpha_1(s, t)x(s, t) + \beta_1(s, t)|u(s, t)|^{r-2}u(s, t) \right) dt ds. \quad (2.3)$$

On the other hand, we have

$$\begin{aligned} \int_a^\tau \int_c^\sigma \frac{\partial^2 x(s, t)}{\partial s \partial t} dt ds &= \int_a^\tau \int_c^\sigma \frac{\partial}{\partial t} \left(\frac{\partial x(s, t)}{\partial s} \right) dt ds \\ &= \int_a^\tau \left[\int_c^\sigma \frac{\partial x(s, t)}{\partial s} \Big|_t dt \right] ds \\ &= \int_a^\tau \frac{\partial x(s, \sigma)}{\partial s} ds - \int_a^\tau \frac{\partial x(s, c)}{\partial s} ds \\ &= x(\tau, \sigma) - x(a, \sigma) - x(\tau, c) + x(a, c) \\ &= x(\tau, \sigma). \end{aligned} \quad (2.4)$$

Hence,

$$x(\tau, \sigma) = \int_a^\tau \int_c^\sigma \left(\alpha_1(s, t)x(s, t) + \beta_1(s, t)|u(s, t)|^{r-2}u(s, t) \right) dt ds, \quad (2.5)$$

and similarly, we have

$$x(\tau, \sigma) = \int_\tau^b \int_\sigma^d \left(\alpha_1(s, t)x(s, t) + \beta_1(s, t)|u(s, t)|^{r-2}u(s, t) \right) dt ds. \quad (2.6)$$

Employing the triangle inequality gives

$$|x(\tau, \sigma)| \leq \int_a^\tau \int_c^\sigma |\alpha_1(s, t)||x(s, t)| dt ds + \int_a^\tau \int_c^\sigma \beta_1(s, t)|u(s, t)|^{r-1} dt ds, \quad (2.7)$$

$$|x(\tau, \sigma)| \leq \int_\tau^b \int_\sigma^d |\alpha_1(s, t)||x(s, t)| dt ds + \int_\tau^b \int_\sigma^d \beta_1(s, t)|u(s, t)|^{r-1} dt ds. \quad (2.8)$$

Summing (2.7) and (2.8), we obtain

$$2|x(\tau, \sigma)| \leq \int_a^\tau \int_c^\sigma |\alpha_1(s, t)||x(s, t)| dt ds + \int_\tau^b \int_\sigma^d \beta_1(s, t)|u(s, t)|^{r-1} dt ds. \quad (2.9)$$

By using Hölder inequality on the second integral of the right side of (2.9) with indices α and γ , we have

$$\begin{aligned}
 & \int_a^b \int_c^d \beta_1(s, t) |u(s, t)|^{\gamma-1} dt ds \\
 &= \int_a^b \int_c^d \beta_1(s, t)^{1/\gamma} \beta_1(s, t)^{1/\alpha} |u(s, t)|^{\gamma-1} dt ds \\
 &\leq \left(\int_a^b \int_c^d \beta_1(s, t) dt ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_1(s, t) |u(s, t)|^{\alpha(\gamma-1)} dt ds \right)^{1/\alpha} \\
 &= \left(\int_a^b \int_c^d \beta_1(s, t) dt ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_1(s, t) |u(s, t)|^\gamma dt ds \right)^{1/\alpha},
 \end{aligned} \tag{2.10}$$

where $(1/\alpha) + (1/\gamma) = 1$.

Therefore, we obtain from (2.9)

$$\begin{aligned}
 2|x(\tau, \sigma)| &\leq \int_a^b \int_c^d |\alpha_1(s, t)| |x(s, t)| dt ds \\
 &+ \left(\int_a^b \int_c^d \beta_1(s, t) dt ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_1(s, t) |u(s, t)|^\gamma dt ds \right)^{1/\alpha}.
 \end{aligned} \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}
 \frac{\partial^2}{\partial s \partial t} (x(s, t) u(s, t)) &= \frac{\partial}{\partial t} \left(\frac{\partial x(s, t)}{\partial s} \cdot u(s, t) + x(s, t) \cdot \frac{\partial u(s, t)}{\partial s} \right) \\
 &= \frac{\partial^2 x(s, t)}{\partial s \partial t} \cdot u(s, t) + \frac{\partial x(s, t)}{\partial s} \cdot \frac{\partial u(s, t)}{\partial t} \\
 &\quad + \frac{\partial x(s, t)}{\partial t} \cdot \frac{\partial u(s, t)}{\partial s} + x(s, t) \cdot \frac{\partial^2 u(s, t)}{\partial s \partial t}.
 \end{aligned} \tag{2.12}$$

Multiplying the first equation of (2.1) by $u(s, t)$ and the second one by $x(s, t)$, adding the result, and noting $(\partial u(s, t)/\partial s)(\partial x(s, t)/\partial t) + (\partial u(s, t)/\partial t)(\partial x(s, t)/\partial s) = 0$, we have

$$\frac{\partial^2}{\partial s \partial t} [x(s, t) u(s, t)] = \beta_1(s, t) |u(s, t)|^\gamma - \beta_2(s, t) |x(s, t)|^\beta. \tag{2.13}$$

Integrating the left side of (2.13) over t from c to d and over s from a to b , respectively, we get

$$\begin{aligned}
 & \int_a^b \int_c^d \frac{\partial^2}{\partial s \partial t} [x(s, t)u(s, t)] dt ds \\
 &= \int_a^b \int_c^d \frac{\partial}{\partial t} \left[\frac{\partial(x(s, t)u(s, t))}{\partial s} \right] dt ds \\
 &= \int_a^b \left[\int_c^d \frac{\partial(x(s, t)u(s, t))}{\partial s} \Big|_t dt \right] ds \\
 &= \int_a^b \frac{\partial(x(s, d)u(s, d))}{\partial s} ds - \int_a^b \frac{\partial(x(s, c)u(s, c))}{\partial s} ds \\
 &= x(b, d)u(b, d) - x(a, d)u(a, d) - x(b, c)u(b, c) + x(a, c)u(a, c).
 \end{aligned} \tag{2.14}$$

Now integrating both sides of (2.13) over t from c to d and over s from a to b , respectively, and noting $x(a, t) = x(b, t) = 0$, we get

$$\int_a^b \int_c^d \beta_1(s, t)|u(s, t)|^\gamma dt ds = \int_a^b \int_c^d \beta_2(s, t)|x(s, t)|^\beta dt ds. \tag{2.15}$$

Substituting equality (2.15) by (2.11), we have

$$\begin{aligned}
 2|x(\tau, \sigma)| &\leq \int_a^b \int_c^d |\alpha_1(s, t)||x(s, t)| dt ds \\
 &+ \left(\int_a^b \int_c^d \beta_1(s, t) dt ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2(s, t)|x(s, t)|^\beta dt ds \right)^{1/\alpha}.
 \end{aligned} \tag{2.16}$$

Noticing that $M = |x(\tau, \sigma)| = \max_{\substack{a < s < b \\ c < t < d}} |x(s, t)| > 0$ and $\beta_2^+(s, t) = \max_{\substack{a < s < b \\ c < t < d}} \{\beta_2(s, t), 0\}$, we obtain

$$2 \leq \int_a^b \int_c^d |\alpha_1(s, t)| dt ds + M^{\beta/\alpha-1} \left(\int_a^b \int_c^d \beta_1(s, t) dt ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2^+(s, t) dt ds \right)^{1/\alpha}. \tag{2.17}$$

The proof is complete. \square

Remark 2.2. Let $x(s, t), u(s, t), \alpha_1(s, t)$, and $\beta_1(s, t)$ change to $x(t), u(t), \alpha_1(t)$, and $\beta_1(t)$ in (2.2), and with suitable changes, (2.2) changes to the following result:

$$2 \leq \int_a^b |\alpha_1(t)| dt + M^{\beta/\alpha-1} \left(\int_a^b \beta_1(t) dt \right)^{1/\gamma} \left(\int_a^b \beta_2^+(t) dt \right)^{1/\alpha}. \quad (2.18)$$

This is just a new Lyapunov-type inequality which was given by Tiryaki et al. [8].

3. The Lyapunov-Type Discrete Inequality for the Two-Dimensional Nonlinear System

$$\begin{aligned} \Delta_1 \Delta_2 x(s, t) &= \alpha_1(s, t)x(s+1, t+1) + \beta_1(s, t)|u(s, t)|^{\gamma-2}u(s, t), \\ \Delta_1 \Delta_2 u(s, t) &= -\beta_2(s, t)|x(s+1, t+1)|^{\beta-2}x(s+1, t+1) - \alpha_1(s, t)u(s, t), \end{aligned} \quad (3.1)$$

where $s, t \in \mathbb{Z}$, Δ_1 denotes the forward difference operator for s , that is, $\Delta_1 x(s, t) = x(s+1, t) - x(s, t)$, and Δ_2 denotes the forward difference operator for t , that is, $\Delta_2 x(s, t) = x(s, t+1) - x(s, t)$. We shall assume the existence of nontrivial solution $(x(s, t), u(s, t))$ of the system (3.1), and furthermore, (3.1) satisfies the following assumptions (i), (ii), and (iii):

- (i) $\gamma > 1, \beta > 1$ are real constants;
- (ii) $\beta_1(s, t), \beta_2(s, t)$ are real-valued functions such that $\beta_1(s, t) > 0$ for all $s, t \in \mathbb{Z}$;
- (iii) $\alpha_1(s, t)$ is a real-valued function for all $s, t \in \mathbb{Z}$.

Theorem 3.1. *Let the hypotheses (i)–(iii) hold. Assume $n_1, m_1, n_2, m_2 \in \mathbb{Z}$ and $n_1 < m_1 - 2, n_2 < m_2 - 2$. If the nonlinear system (3.1) has a real solution $(x(s, t), u(s, t))$ such that $x(n_1, t) = x(m_1, t) = x(s, n_2) = x(s, m_2) = 0$ for all $(s, t) \in [n_1, m_1] \times [n_2, m_2]$, and $\Delta_1 x(s, t+1) \cdot \Delta_2 u(s, t) + \Delta_2 x(s+1, t) \cdot \Delta_1 u(s, t) = 0$ and $x(s, t)$ is not identically zero on $[n_1, m_1] \times [n_2, m_2]$, then*

$$2 \leq \sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} |\alpha_1(s, t)| + M^{\beta/\alpha-1} \left(\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) \right)^{1/\gamma} \left(\sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} \beta_2^+(s, t) \right)^{1/\alpha}, \quad (3.2)$$

where $(1/\alpha) + (1/\gamma) = 1, M = |x(\tau, \sigma)| = \max_{\substack{n_1+1 \leq s < m_1-1 \\ n_2+1 \leq t < m_2-1}} |x(s, t)|$, and $\beta_2^+(s, t) = \max_{\substack{n_1+1 \leq s < m_1-1 \\ n_2+1 \leq t < m_2-1}} \{\beta_2(s, t), 0\}$.

Proof. Let $(x(s, t), u(s, t))$ be nontrivial real solution of system (3.1) such that $x(n_1, t) = x(m_1, t) = x(s, n_2) = x(s, m_2) = 0$ and $x(s, t)$ is not identically zero on $[n_1, m_1] \times [n_2, m_2]$.

Then multiplying the first equation of (3.1) by $u(s, t)$ and the second one by $x(s + 1, t + 1)$, adding the result, and noting $\Delta_1 x(s, t + 1) \cdot \Delta_2 u(s, t) + \Delta_2 x(s + 1, t) \cdot \Delta_1 u(s, t) = 0$, and

$$\begin{aligned}
 & \Delta_1 \Delta_2 [x(s, t) u(s, t)] \\
 &= \Delta_2 ((x(s + 1, t) - x(s, t)) u(s, t) + x(s + 1, t) (u(s + 1, t) - u(s, t))) \\
 &= \Delta_2 ((x(s + 1, t) - x(s, t)) u(s, t)) + \Delta_2 (x(s + 1, t) (u(s + 1, t) - u(s, t))) \\
 &= (x(s + 1, t + 1) - x(s, t + 1) - (x(s + 1, t) - x(s, t))) u(s, t) \\
 &\quad + (x(s + 1, t + 1) - x(s, t + 1)) (u(s, t + 1) - u(s, t)) \\
 &\quad + (x(s + 1, t + 1) - x(s + 1, t)) (u(s + 1, t) - u(s, t)) \\
 &\quad + x(s + 1, t + 1) (u(s + 1, t + 1) - u(s, t + 1) - (u(s + 1, t) - u(s, t))) \\
 &= (\Delta_1 \Delta_2 x(s, t)) u(s, t) + \Delta_1 x(s, t + 1) \Delta_2 u(s, t) \\
 &\quad + \Delta_2 x(s + 1, t) \Delta_1 u(s, t) + x(s + 1, t + 1) (\Delta_1 \Delta_2 u(s, t)),
 \end{aligned} \tag{3.3}$$

we have

$$\Delta_1 \Delta_2 [x(s, t) u(s, t)] = \beta_1(s, t) |u(s, t)|^\gamma - \beta_2(s, t) |x(s + 1, t + 1)|^\beta. \tag{3.4}$$

Summing the left side of (3.4) over t from n_2 to $m_2 - 1$ and over s from n_1 to $m_1 - 1$, respectively, we have

$$\begin{aligned}
 & \sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \Delta_1 \Delta_2 (x(s, t) u(s, t)) \\
 &= \sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} (x(s + 1, t + 1) u(s + 1, t + 1) - x(s + 1, t) u(s + 1, t) \\
 &\quad - (x(s, t + 1) u(s, t + 1) - x(s, t) u(s, t))) \\
 &= \sum_{s=n_1}^{m_1-1} (x(s + 1, m_2) u(s + 1, m_2) - x(s, m_2) u(s, m_2) \\
 &\quad - (x(s + 1, n_2) u(s + 1, n_2) - x(s, n_2) u(s, n_2))) \\
 &= x(m_1, m_2) u(m_1, m_2) - x(n_1, m_2) u(n_1, m_2) - x(m_1, n_2) u(m_1, n_2) \\
 &\quad + x(n_1, n_2) u(n_1, n_2).
 \end{aligned} \tag{3.5}$$

Summing both sides of (3.4) over t from n_2 to $m_2 - 1$ and over s from n_1 to $m_1 - 1$, respectively, and noting $x(n_1, t) = x(m_1, t) = 0$, we obtain

$$\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) |u(s, t)|^\gamma = \sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_2(s, t) |x(s + 1, t + 1)|^\beta. \tag{3.6}$$

Noticing that $x(m_1, t) = x(s, m_2) = 0$ and $\beta_2^+(s, t) = \max_{\substack{n_1+1 < s < m_1-1 \\ n_2+1 < t < m_2-1}} \{\beta_2(s, t), 0\}$, we have

$$\begin{aligned} \sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) |u(s, t)|^\gamma &= \sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} \beta_2(s, t) |x(s+1, t+1)|^\beta \\ &\leq \sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} \beta_2^+(s, t) |x(s+1, t+1)|^\beta. \end{aligned} \quad (3.7)$$

Choose $(\tau, \sigma) \in [n_1 + 1, m_1 - 1] \times [n_2 + 1, m_2 - 1]$ such that $M = |x(\tau, \sigma)| = \max_{\substack{n_1+1 < s < m_1-1 \\ n_2+1 < t < m_2-1}} |x(s, t)|$. Hence $M = |x(\tau, \sigma)| > 0$. Summing the first equation of (3.1) over t from n_2 to $\sigma - 1$ and over s from n_1 to $\tau - 1$, respectively, we obtain

$$\sum_{s=n_1}^{\tau-1} \sum_{t=n_2}^{\sigma-1} \Delta_1 \Delta_2 x(s, t) = \sum_{s=n_1}^{\tau-1} \sum_{t=n_2}^{\sigma-1} \alpha_1(s, t) x(s+1, t+1) + \sum_{s=n_1}^{\tau-1} \sum_{t=n_2}^{\sigma-1} \beta_1(s, t) |u(s, t)|^{\gamma-2} u(s, t). \quad (3.8)$$

Considering the left side of (3.8) and noting $x(n_1, t) = x(s, n_2) = 0$ for all $(s, t) \in [n_1, m_1] \times [n_2, m_2]$, we have

$$\begin{aligned} \sum_{s=n_1}^{\tau-1} \sum_{t=n_2}^{\sigma-1} \Delta_1 \Delta_2 x(s, t) &= \sum_{s=n_1}^{\tau-1} \left(\sum_{t=n_2}^{\sigma-1} (x(s+1, t+1) - x(s+1, t) - (x(s, t+1) - x(s, t))) \right) \\ &= \sum_{s=n_1}^{\tau-1} (x(s+1, \sigma) - x(s, \sigma) - (x(s+1, n_2) - x(s, n_2))) \\ &= x(\tau, \sigma) - x(n_1, \sigma) - x(\tau, n_2) + x(n_1, n_2) \\ &= x(\tau, \sigma). \end{aligned} \quad (3.9)$$

Hence,

$$x(\tau, \sigma) = \sum_{s=n_1}^{\tau-1} \sum_{t=n_2}^{\sigma-1} \alpha_1(s, t) x(s+1, t+1) + \sum_{s=n_1}^{\tau-1} \sum_{t=n_2}^{\sigma-1} \beta_1(s, t) |u(s, t)|^{\gamma-2} u(s, t), \quad (3.10)$$

and similarly, we have

$$x(\tau, \sigma) = \sum_{s=\tau}^{m_1-2} \sum_{t=\sigma}^{m_2-2} \alpha_1(s, t) x(s+1, t+1) + \sum_{s=\tau}^{m_1-1} \sum_{t=\sigma}^{m_2-1} \beta_1(s, t) |u(s, t)|^{\gamma-2} u(s, t). \quad (3.11)$$

Employing the triangle inequality gives

$$|x(\tau, \sigma)| \leq \sum_{s=n_1}^{\tau-1} \sum_{t=n_2}^{\sigma-1} |\alpha_1(s, t)| |x(s+1, t+1)| + \sum_{s=n_1}^{\tau-1} \sum_{t=n_2}^{\sigma-1} \beta_1(s, t) |u(s, t)|^{\gamma-1}, \quad (3.12)$$

$$|x(\tau, \sigma)| \leq \sum_{s=\tau}^{m_1-2} \sum_{t=\sigma}^{m_2-2} |\alpha_1(s, t)| |x(s+1, t+1)| + \sum_{s=\tau}^{m_1-1} \sum_{t=\sigma}^{m_2-1} \beta_1(s, t) |u(s, t)|^{\gamma-1}. \quad (3.13)$$

Summing (3.12) and (3.13), we obtain

$$2|x(\tau, \sigma)| \leq \sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} |\alpha_1(s, t)| |x(s+1, t+1)| + \sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) |u(s, t)|^{\gamma-1}. \quad (3.14)$$

On the other hand, using Hölder inequality on the second sum of the right side of (3.14) with indices α and γ , we have

$$\begin{aligned} \sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) |u(s, t)|^{\gamma-1} &= \sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t)^{1/\gamma} \beta_1(s, t)^{1/\alpha} |u(s, t)|^{\gamma-1} \\ &\leq \left(\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) \right)^{1/\gamma} \left(\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) |u(s, t)|^{\alpha(\gamma-1)} \right)^{1/\alpha} \\ &= \left(\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) \right)^{1/\gamma} \left(\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) |u(s, t)|^\gamma \right)^{1/\alpha}, \end{aligned} \quad (3.15)$$

where $(1/\alpha) + (1/\gamma) = 1$. Therefore, from (3.7) and (3.10), we obtain

$$\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) |u(s, t)|^{\gamma-1} \leq \left(\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) \right)^{1/\gamma} \left(\sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} \beta_2^+(s, t) |x(s+1, t+1)|^\beta \right)^{1/\alpha}. \quad (3.16)$$

Substituting (3.16) to (3.14), we have

$$\begin{aligned} 2|x(\tau, \sigma)| &\leq \sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} |\alpha_1(s, t)| |x(s+1, t+1)| \\ &\quad + \left(\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) \right)^{1/\gamma} \left(\sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} \beta_2^+(s, t) |x(s+1, t+1)|^\beta \right)^{1/\alpha}. \end{aligned} \quad (3.17)$$

Noticing that $M = |x(\tau, \sigma)| = \max_{\substack{n_1+1 < s < m_1-1 \\ n_2+1 < t < m_2-1}} |x(s, t)| > 0$, we get

$$2 \leq \sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} |\alpha_1(s, t)| + M^{\beta/\alpha-1} \left(\sum_{s=n_1}^{m_1-1} \sum_{t=n_2}^{m_2-1} \beta_1(s, t) \right)^{1/\gamma} \left(\sum_{s=n_1}^{m_1-2} \sum_{t=n_2}^{m_2-2} \beta_2^+(s, t) \right)^{1/\alpha}. \quad (3.18)$$

This completes the proof. \square

Remark 3.2. Let $x(s, t), u(s, t), \alpha_1(s, t)$, and $\beta_1(s, t)$ change to $x(t), u(t), \alpha_1(t)$, and $\beta_1(t)$ in (3.2) and with suitable changes, (3.2) changes to the following result:

$$2 \leq \sum_{t=n}^{m-2} |\alpha_1(t)| + M^{\beta/\alpha-1} \left(\sum_{t=n}^{m-1} \beta_1(t) \right)^{1/\gamma} \left(\sum_{t=n}^{m-2} \beta_2^+(t) \right)^{1/\alpha}. \quad (3.19)$$

This is just a new Lyapunov-type inequality which was given by Ünal et al. [2].

4. An application

Two-dimensional Emden-Fowler-type equation

$$\frac{\partial}{\partial s \partial t} \left(r(s, t) \left| \frac{\partial x(x, t)}{\partial s \partial t} \right|^{\alpha-2} \frac{\partial x(x, t)}{\partial s \partial t} \right) + q(s, t) |x(s, t)|^{\beta-2} x(s, t) = 0, \quad (4.1)$$

where $\alpha > 1$ is a constant, $r(s, t)$ and $q(s, t)$ are real functions, and $r(s, t) > 0$ for all $(s, t) \in \mathbb{R} \times \mathbb{R}$.

Consider the following special case of system (2.1), which is an equivalent system for the two-dimensional Emden-Fowler-type equation (4.1)

$$\begin{aligned} \frac{\partial x^2(s, t)}{\partial s \partial t} &= \beta_1(s, t) |u(s, t)|^{\gamma-2} u(s, t), \\ \frac{\partial u^2(s, t)}{\partial s \partial t} &= -\beta_2(s, t) |x(s, t)|^{\beta-2} x(s, t), \end{aligned} \quad (4.2)$$

where $\beta_1(s, t) = r(s, t)^{1-\gamma}$ and $\beta_2(s, t) = q(s, t)$.

Obviously Theorem 2.1 for the two-dimensional nonlinear system (2.1) with $\alpha_1(s, t) \equiv 0$ is satisfied for system (4.2). Therefore, we have

$$2 \leq M^{\beta/\alpha-1} \left(\int_a^b \int_c^d \beta_1(s, t) dt ds \right)^{1/\gamma} \left(\int_a^b \int_c^d \beta_2^+(s, t) dt ds \right)^{1/\alpha}. \quad (4.3)$$

A nontrivial solution $(x(s, t), u(s, t))$ of system (4.2) defined on $[s_0, \infty) \times [t_0, \infty)$ is said to be *proper* if and only if

$$\sup\{|x(s, t)| + |u(s, t)| : a \leq s < \infty, c \leq t < \infty\} > 0, \quad (4.4)$$

for any $a \geq s_0, c \geq t_0$. A proper solution $(x(s, t), u(s, t))$ of system (4.2) is called *weakly oscillatory* if and only if at least one component has a sequence of zeros tending to $+\infty$.

Theorem 4.1. *If $|x(\tau, \sigma)| = \max\{|x(s, t)| : a < s < b, c < t < d\}$, where $a > s_0, c > t_0$ and $s_0, t_0, a, b, c, d \in \mathbb{R}$, $u(\tau, t)$ is bounded on $[t_0, \infty)$ and $u(s, \sigma)$ is bounded on $[s_0, \infty)$,*

$$\int_a^\infty \int_c^\infty \beta_1(s, t) dt ds < \infty, \quad \int_a^\infty \int_c^\infty |\beta_2(s, t)| dt ds < \infty, \quad (4.5)$$

then every weakly oscillatory proper solution of (4.2) is bounded on $I = [s_0, \infty) \times [t_0, \infty)$.

Proof. Let $(x(s, t), u(s, t))$ be any nontrivial weakly oscillatory proper solution of nonlinear system (4.2) on $I = [s_0, \infty) \times [t_0, \infty)$ such that $x(s, t)$ has a sequence of zeros tending to $+\infty$. Suppose to the contrary that $\limsup |x(s, t)| = \infty$; then given any positive number M_0 , we can find positive numbers S_0 and T_0 such that $|x(s, t)| > M_0$ for all $s > S_0, t > T_0$. Since $x(s, t)$ is an oscillatory solution, there exist $(a, b) \times (c, d) \in \mathbb{R} \times \mathbb{R}$ with $a > S_0, c > T_0$ such that $x(a, t) = x(b, t) = x(s, c) = x(s, d) = 0$ and $|x(s, t)| > 0$ on $(a, b) \times (c, d)$. Choose (τ, σ) in $(a, b) \times (c, d)$ such that $M = |x(\tau, \sigma)| = \max\{|x(s, t)| : a < s < b, c < t < d\} > M_0$; in view of (4.5), we can choose S_0 and T_0 large enough such that for every $a \geq S_0, c \geq T_0$,

$$\int_a^\infty \int_c^\infty \beta_1(s, t) dt ds < M^{-(\beta-\alpha)/(\alpha-1)}, \quad \int_a^\infty \int_c^\infty |\beta_2(s, t)| dt ds < 1. \quad (4.6)$$

Taking α th power of both sides of (4.3) and combining (4.6), we obtain

$$\begin{aligned} 2^\alpha &\leq M^{\beta-\alpha} \left(\int_a^b \int_c^d \beta_1(s, t) dt ds \right)^{\alpha-1} \left(\int_a^b \int_c^d \beta_2^+(s, t) dt ds \right) \\ &\leq M^{\beta-\alpha} \left(\int_a^\infty \int_c^\infty \beta_1(s, t) dt ds \right)^{\alpha-1} \left(\int_a^\infty \int_c^\infty |\beta_2(s, t)| dt ds \right) \\ &< M^{\beta-\alpha} M^{-\beta+\alpha} = 1, \end{aligned} \quad (4.7)$$

where $\alpha > 1$ and $\beta_2^+(s, t) \leq |\beta_2(s, t)|$.

This contradiction shows that $|x(s, t)|$ is bounded on $I = [s_0, \infty) \times [t_0, \infty)$. Therefore, there exists a positive constant K such that $|x(s, t)| \leq K$ for all $(s, t) \in I$.

On the other hand, integrating the second equation of system (4.2) over t from σ to t and over s from σ to s , respectively, we obtain

$$u(s, t) - u(\tau, t) - u(s, \sigma) + u(\tau, \sigma) = \int_\sigma^s \int_\tau^t -\beta_2(s, t) |x(s, t)|^{\beta-2} x(s, t) dt ds. \quad (4.8)$$

Notice that $u(\tau, t)$ is bounded on $[t_0, \infty)$, $u(s, \sigma)$ is bounded on $[s_0, \infty)$, and in view of triangle inequality, we have

$$\begin{aligned} |u(s, t)| &\leq |u(\tau, t) + u(s, \sigma) - C| + \int_{\sigma}^s \int_{\tau}^t |\beta_2(s, t)| |x(s, t)|^{\beta-1} dt ds \\ &\leq |u(\tau, t) + u(s, \sigma) - C| + K^{\beta-1} \int_{\sigma}^{\infty} \int_{\tau}^{\infty} |\beta_2(s, t)| dt ds, \end{aligned} \quad (4.9)$$

where $C = u(\tau, \sigma)$ is a constant.

Equation (4.9) implies that $|u(s, t)|$ is bounded on $I = [s_0, \infty) \times [t_0, \infty)$ since $\int_{\tau}^{\infty} \int_{\sigma}^{\infty} |\beta_2(s, t)| dt ds < \infty$. It follows from

$$\limsup \{|x(s, t)| + |u(s, t)|\} \leq \limsup |x(s, t)| + \limsup |u(s, t)| \quad (4.10)$$

that $\limsup \{|x(s, t)| + |u(s, t)|\}$ is bounden on $I = [s_0, \infty) \times [t_0, \infty)$.

This completes the proof. \square

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References

- [1] A. M. Liapunov, "Problème général de la stabilité du mouvement," *Annales de la Faculté des Sciences de Toulouse*, vol. 2, pp. 27–247, 1907, French translation of a Russian paper dated 1893.
- [2] M. Ünal, D. Çakmak, and A. Tiriyaki, "A discrete analogue of Lyapunov-type inequalities for nonlinear systems," *Computers & Mathematics with Applications*, vol. 55, no. 11, pp. 2631–2642, 2008.
- [3] H. Ito, "A degree of flexibility in Lyapunov inequalities for establishing input-to-state stability of interconnected systems," *Automatica*, vol. 44, no. 9, pp. 2340–2346, 2008.
- [4] A. Cañada, J. A. Montero, and S. Villegas, "Lyapunov inequalities for partial differential equations," *Journal of Functional Analysis*, vol. 237, no. 1, pp. 176–193, 2006.
- [5] L. Jiang and Z. Zhou, "Lyapunov inequality for linear Hamiltonian systems on time scales," *Journal of Mathematical Analysis and Applications*, vol. 310, no. 2, pp. 579–593, 2005.
- [6] G. Sh. Guseinov and B. Kaymakçalan, "Lyapunov inequalities for discrete linear Hamiltonian systems," *Computers & Mathematics with Applications*, vol. 45, no. 6–9, pp. 1399–1416, 2003.
- [7] B. G. Pachpatte, "Lyapunov type integral inequalities for certain differential equations," *Georgian Mathematical Journal*, vol. 4, no. 2, pp. 139–148, 1997.
- [8] A. Tiriyaki, M. Ünal, and D. Çakmak, "Lyapunov-type inequalities for nonlinear systems," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 1, pp. 497–511, 2007.